Overview:

- Today we'll look at representations of complex semisimple Lie Algebras.

Important to know that:

- Every finite-dimensional representation is completely reducible and can therefore be written in the form

$$V = \bigoplus n_i V_i ,$$

where V_i are irreducible representations and $n_i \in \mathbb{Z}_+$ are the multiplicities.

  - Benefit: This simplifies the study of these representations to the classification of irreducible representations and finding a way to determine for some representation V, the multiplicities n_i.

- Note: $\mathfrak{g}$ is a complex finite-dimensional semisimple lie algebra.
  - We can fix a choice of a Cartan Subalgebra and thus a root decomposition $\mathfrak{g} = \mathfrak{h} \bigoplus_R \mathfrak{g}_\alpha$.

Every representation considered in this lecture will be complex and unless specified finite-dimensional.

Important Starting Notes

Two Important Definitions:

Def (R-module)

Let R be a ring (always assumed commutative, with unity).
An R-Module M is a triple $(M, +, \cdot)$ where $(M,+)$ is an abelian group, and there is a function $R \times M \to M$ whose value at (r,m) is denoted rm or $r \cdot m$ such that:

(1) $\forall r,s \in R \ \forall m \in M \ r(sm) = (rs)m$

(2) $\forall r,s \in R \ \forall m \in M \ (r+s)m = rm + sm$

(3) $\forall r \in R \ \forall m,n \in M \ r(m+n) = rm + rn$

(4) $\forall m \in M \ 1 \cdot m = m$

This can be thought of as an analogue of vector spaces for more general rings.

Def (Ideal)

Let R be a ring. $I \subseteq R$ is an ideal if:

1) $(I,+) \leq (R,+)$ (I is an additive subgroup of R)

2) $\forall a \in I \ \forall r \in R \ ar \in I$ $(RI \subseteq I)$

Enough to check if $I \neq \emptyset$, I is closed under addition, and this second property that $\forall a \in I \ \forall r \in R \ ar \in I$ holds.

Weight Decomposition and Characters (Kirillov Ch 8.1)

The strategy to study representations of $\mathfrak{g}$ (Complex Semisimple Lie Algebra)

- Decompose the representation into eigenspaces for the Cartan Subalgebra.

__Def__ Let V be a representation of $\mathfrak{g}$. A vector $v \in V$ is called a __vector of weight λ__ $\in \mathfrak{h}^*$ if:

$\forall h \in \mathfrak{h}$, one has $hv = \langle \lambda, h \rangle v$.

The space of all vectors of weight λ is called the __weight space__ and denoted $V[\lambda]$:

$$V[\lambda] = \{v \in V \mid hv = \langle \lambda, h \rangle v \ \forall h \in \mathfrak{h}\}$$

If $V[\lambda] \neq \{0\}$, then λ is called a __weight of V__.

__The set of all weights of V__ is denoted by $P(V)$:

$$P(V) = \{\lambda \in \mathfrak{h}^* \mid V[\lambda] \neq \{0\}\}.$$

- From linear algebra, we can observe that vectors of different weights are in fact linearly independent

$\Longrightarrow$ $P(V)$ is finite for a finite-dimensional representation.

- __Reason:__ The weights can almost be thought of as eigenvalues. Eigenvectors that correspond to different eigenvalues will be linearly independent.

Thm | Every finite dimensional representation of $\mathfrak{g}$ admits a weight decomposition: $V = \bigoplus_{\lambda \in P(V)} V[\lambda]$

Proof. Let $\alpha \in R$ be a root. Consider the corresponding $\mathfrak{sl}(2, \mathbb{C})$ subalgebra in $\mathfrak{g}$ generated by $e_\alpha, f_\alpha, h_\alpha$ as in Lemma 6.42. Considering V is a module over this $\mathfrak{sl}(2, \mathbb{C})$ and using the results of Section 4.8, we see that h_α is a diagonalizable operator in V. Since elements $h_\alpha, \alpha \in R$, span $\mathfrak{h}$, and the sum of the commuting diagonalizable operators is diagonalizable, we see that any $h \in \mathfrak{h}$ is diagonalizable. Since $\mathfrak{h}$ is commutative, all of them can be diagonalized simultaneously, which gives the weight decomposition.

Since weights of $\mathfrak{sl}(2, \mathbb{C})$ must be integer, we see that for any weight λ of V, we must have $\langle \lambda, h_\alpha \rangle \in \mathbb{Z}$, which by definition implies that $\lambda \in P$. □

Kirillov pg. 164 Thm 8.2

Highest Weight Representations and Verma Modules:

- In order to study irreducible representations, we will begin by first discussing a class of representations which are generated by a single vector.
- Considering infinite-dimensional representations is a useful tool in order to aid our understanding of finite-dimensional representations.

Our Lie Algebra $\mathfrak{g}$ can be decomposed in the following way:

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_\pm = \bigoplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$.

Def| A nonzero representation V (potentially infinite dimensional) of $\mathfrak{g}$ is called a highest weight representation if it is generated by a vector $v \in V[\lambda]$ s.t.

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \qquad \mathfrak{n}_\pm = \bigoplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$$

$\mathfrak{n}_-$: Direct sum of root spaces consisting of negative roots

$\mathfrak{n}_+$: Direct sum of root spaces consisting of positive roots

$\mathfrak{h}$: Cartan Subalgebra

Thm 8.10| Every irreducible finite-dimensional representation of $\mathfrak{g}$ is a highest weight representation. Kirilov pg. 167

Proof. Let $P(V)$ be the set of weights of V. Let $\lambda \in P(V)$ be such that for all $\alpha \in R_+$, $\lambda + \alpha \notin P(V)$. Such a λ exists: for example, we can take $h \in \mathfrak{h}$ such that $\langle h, \alpha \rangle > 0$ for all $\alpha \in R_+$, and then consider $\lambda \in P(V)$ such that $\langle h, \lambda \rangle$ is maximal possible.

Now let $v \in V[\lambda]$ be a non-zero vector. Since $\lambda + \alpha \notin P(V)$, we have $e_\alpha v = 0$ for any $\alpha \in R_+$. Consider the subrepresentation $V' \subset V$ generated by v. By definition, V' is a highest weight representation. On the other hand, since V is irreducible, one has $V' = V$. □

Note: It is possible that we have many $\boxed{\text{non-isomorphic}}$ highest weight representations with the same highest weight. However, in any highest representation with highest weight vector $v_\lambda \in V[\lambda]$, the following holds:

1) $h v_\lambda = \langle h, \lambda \rangle v \quad \forall h \in \mathfrak{h}$

2) $x v_\lambda = 0 \quad \forall x \in \mathfrak{n}_+$

Def) The universal highest weight representation M_λ, generated by a vector v_λ satisfying

1) $hv_\lambda = \langle h,\lambda\rangle v \quad \forall h\in\mathfrak{h}$
2) $xv_\lambda = 0 \quad \forall x\in n_+$

$\}$ Same conditions as above *

and no other relations.

$$M_\lambda := Ug/I_\lambda.$$

I_λ is the left ideal in Ug generated vectors $e\in n_+$ and $(h-\langle h,\lambda\rangle)$, $h\in\mathfrak{h}$.

This module is called the verma models.

- Quite important to representation theory.

Define the Borel subalgebra b by

$$b = \mathfrak{h}\oplus n_+$$

* conditions define a one-dimensional representation of b which we will denote $\mathbb{C}_\lambda$.

$$M_\lambda := Ug \otimes_{Ub} \mathbb{C}_\lambda.$$

__Lemma__ If V is a highest weight representation with highest weight λ, then $V \simeq M_\lambda / W$ for some submodule $W \subset M_\lambda$.

- We can thus reinterpret the study of highest weight representations as the study of submodules in Verma modules.

__Important facts:__

Let V be a highest weight representation with highest weight λ (not necessarily finite-dimensional)

1) $\forall v \in M_\lambda \quad v = uv_\lambda, u \in U_{n_-}$.

In other words, the map

$$\begin{aligned} U_{n_-} &\to M_\lambda \\ u &\mapsto uv_\lambda \end{aligned}$$

is surjective.

2) V admits a weight decomposition: $V = \bigoplus_{\mu \leq \lambda} V[\mu]$, with finite-dimensional weight subspaces.

3) $\dim M_\lambda[\lambda] = 1$.

Lemma: Let $\mathfrak{h}$ be a commutative, finite-dimensional Lie Algebra and M a module over $\mathfrak{h}$ (not necessarily finite-dimensional) which admits a weight decomposition with finite-dimensional weight spaces:

$$M = \bigoplus M[\lambda], \quad M[\lambda] = \{v \mid hv = \langle h, \lambda \rangle v\}$$

Then any submodule, quotient of M also admits a weight decomposition.

Corrolary: In any highest weight representation, there is a unique highest weight and it is unique up to a scalar highest weight vector.

Pf] This is clear since if λ, μ are highest weights $\Rightarrow \lambda \leq \mu$ since μ is a highest weight and $\mu \leq \lambda$ since λ is a highest weight.

Hence, $\mu = \lambda$. $\square$

Classification of irreducible finite-dimensional representations

Next goal: Classify all irreducible finite-dimensional representations.

Since every irreducible finite-dimensional representation of $\mathfrak{g}$ is a highest weight representation, we can rephrase this goal to classifying all highest weight representations which are finite-dimensional and irreducible.

Thm 8.18 $\forall \lambda \in \mathfrak{h}^*$ $\exists$ irreducible highest weight representation with highest weight λ, which is unique up to isomorphism. We denote this representation as L_λ.

Pf All highest weight representations with highest weight λ are of the form M_λ / W for some $W \subset M_\lambda$.

M_λ / W is irreducible $\Leftrightarrow$ W is a maximal proper subrepresentation

It suffices to show that M_λ has a unique maximal proper submodule.

Since every proper submodule $W \subset M_\lambda$ admits a weight decomposition and $W[\lambda]=0$.
(Note: If this were not the case, then $W[\lambda] = M_\lambda[\lambda] \Rightarrow W = M_\lambda$)

Let J_λ be the sum of all submodules $W \subset M_\lambda$ s.t. $W[\lambda]=0$.
Since it contains every other proper submodule of M_λ, it is the unique maximal proper submodule of M_λ.

As such, $L_\lambda = M_\lambda / J_\lambda$ is the unique irreducible highest-weight module with highest weight λ. $\square$

Exs
For $g = sl(2, \mathbb{C})$, if $\lambda \in \mathbb{Z}_+$, $L_\lambda = V_\lambda$ is the finite-dimensional irreducible module of dimension $\lambda + 1$ and $L_\lambda = M_\lambda$ for $\lambda \notin \mathbb{Z}_+$.

- Since every finite-dimensional representation is a highest weight representation, we get that

Corollary 8.20 | Every irreducible finite-dimensional representation V is isomorphic to one L_λ.

To classify all irreducible finite-dimensional representations of g, we first need to find which L_λ are finite-dimensional.

Def | For a weight $\lambda \in \mathfrak{h}^*$ is called <u>dominant integral</u>, $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_+ \; \forall \alpha \in R_+$.

$P_+ = \{\text{All dominant integral weights}\}$.

Alternate Def | A weight $\lambda \in \mathfrak{h}^*$ is called <u>dominant integral</u> if $\dfrac{2(\alpha, \lambda)}{(\alpha, \alpha)} \in \mathbb{N} \; \forall \alpha \in R_+$.

Thm (8.23) Irreducible highest weight representation L_λ is finite dimensional $\Longleftrightarrow \lambda \in P_+$.

Proof in Kirillov pg. 172–173 Thm. 8.23.

Sketch of Pf:

"$\Longrightarrow$": Suppose $\dim L_\lambda = n < \infty$.

- Let $\alpha \in R_+$ be a positive root.
- $sl(2,\mathbb{C})_\alpha = \langle e_\alpha, f_\alpha, h_\alpha \rangle$ ← subalgebra in $\mathfrak{g}$ which is generated by $e_\alpha, f_\alpha, h_\alpha$.
- The highest weight vector $v_\lambda \in L_\lambda$ satisfies $e_\alpha v_\lambda = 0$ and $h_\alpha v_\lambda = \lambda(h_\alpha) v_\lambda = \frac{2(\alpha,\lambda)}{(\alpha,\alpha)} v_\lambda$.
- However, in a finite-dimensional representation of $sl(2,\mathbb{C})$, the highest weight is a non-negative integer.
- Hence, $\frac{2(\alpha,\lambda)}{(\alpha,\alpha)} \in \mathbb{N} \Longrightarrow \lambda \in P_+$.

"$\Longleftarrow$": You can apply the following theorem

Let $\lambda \in P_+$ and $\tilde{L}_\lambda := M_\lambda / \sum_i M_i$, where M_i is a proper submodule of M_λ.

Then, $\tilde{L}_\lambda$ is finite-dimensional.

(Proof of Thm: Kirillov Chapter 8.4 [Entire Section])

$\square$

Corollary: $\forall \lambda \in P_+$, L_λ is an irreducible finite-dimensional representation.

These representations are pairwise non-isomorphic. Every irreducible finite-dimensional representation is isomorphic to one of them.

The last Theorem we just proved and the fact that every irreducible representation of $\mathfrak{g}$ is a highest weight representation. $\Rightarrow$ This corollary

Thm (Highest-Weight Theorem)

(1) Every irreducible (finite-dimensional) representation has a highest weight.

(2) The highest weight is always a dominant algebraically integral element.

(3) Two irreducible representations with the same highest weight are isomorphic

(4) Every dominant algebraically integral element is the highest weight of an irreducible representation.

Already proved most of these.

Thm 8.10 for (1)

Cor 8.20 (2)

Thm 8.18 for (3)

Thm 8.23 for (4)

Related to these Theorems and Corollaries